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ON THE EXISTENCE OF $(v, 7, 1)$-PERFECT MENDELSOHN DESIGNS

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Let v, k, and λ be positive integers. A (v, k, λ)-Mendelsohn design (briefly (v, k, λ)-MD) is a pair $(X, \mathcal{B})$, where X is a v-set (of *points*) and $\mathcal{B}$ is a collection of cyclically ordered k-subsets of X (called *blocks*) such that every ordered pair of points of X are consecutive in exactly λ of the blocks of $\mathcal{B}$. If for all $t = 1, 2, \ldots, k-1$, every ordered pair of points of X are t-apart in exactly λ of the blocks of $\mathcal{B}$, then the (v, k, λ)-MD is called a *perfect* design and denoted briefly by (v, k, λ)-PMD. A necessary condition for the existence of a $(v, 7, 1)$-PMD is $v \equiv 0$ or $1 \pmod 7$. We show that this condition is sufficient for all $v \geq 2136$, with at most 104 possible exceptions below this value. This result is established, for the most part, by means of a result on pairwise balanced designs (PBDs) which is of interest in its own right. If Q^* denotes the set of all prime powers congruent to 0 or 1 modulo 7, then it is shown that a PBD $B(Q^*, 1; v)$ exists for all integers $v \geq 2136$, where $v \equiv 0$ or $1 \pmod 7$, with at most 104 possible exceptions below this value.

1. Introduction

The concept of a perfect cyclic design was introduced by Mendelsohn [16]. This concept was further studied in a subsequent paper [4], where the notion of resolvability was discussed and associations made with certain classes of quasigroups and orthogonal arrays with interesting conjugacy properties. A further development of the concept was made by Hsu and Keedwell [10], where the designs were called Mendelsohn designs and associated with complete mappings and near complete mappings. In what follows, we shall adapt the terminology and notation of Hsu and Keedwell and present the following definitions involving the concept of Mendelsohn designs.

Definition 1.1. A set of k distinct elements $\{a_1, a_2, \ldots, a_k\}$ is said to be cyclically ordered by $a_1 < a_2 < \cdots < a_k < a_1$ and the pair a_i, a_{i+t} are said to be t-apart in a cyclic k-tuple $(a_1, a_2, \ldots, a_k)$ where $i + t$ is taken modulo k.

Definition 1.2. Let v, k, and λ be positive integers. A (v, k, λ)-*Mendelsohn design* (briefly (v, k, λ)-MD) is a pair $(X, \mathcal{B})$ where X is a v-set (of *points*) and $\mathcal{B}$

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is a collection of cyclically ordered k-subsets of X (called *blocks*) such that every ordered pair of points of X are consecutive in exactly λ of the blocks of $\mathcal{B}$. If for all $t = 1, 2, \ldots, k-1$, every ordered pair of points of X are t-apart in exactly λ of the blocks of $\mathcal{B}$, then the (v, k, λ)-MD is called a *perfect* design and denoted briefly by (v, k, λ)-PMD.

We wish to remark that a (v, k, λ)-MD is equivalent to the decomposition of the complete directed multigraph λK_v^* on v vertices into k-circuits. Since the number of blocks in a (v, k, λ)-MD is $\lambda v(v-1)/k$, then a necessary condition for existence is $\lambda v(v-1) \equiv 0 \pmod k$. This condition is known to be sufficient in many cases, but certainly not in all. For example, no design exists when $\lambda = 1$ and $v = k = 4$ or $v = k = 6$ or $v = 6$ and $k = 3$.

In this paper, we shall be concerned mainly with the existence of $(v, 7, 1)$-PMDs, where a necessary condition for existence is $v \equiv 0$ or $1 \pmod 7$. We show that this condition is sufficient for all $v \geqslant 2136$, with at most 104 possible exceptions below this value. For additional results on Mendelsohn designs, the interested reader is referred to [1–4, 10–11, 14–16].

The main result of this paper will be established, for the most part, by means of a result relating to pairwise balanced designs (PBDs) and which is of interest in its own right.

Definition 1.3. Let K be a set of positive integers. A *pairwise balanced design* (PBD) of index unity $B(K, 1; v)$ is a pair $(X, \mathcal{B})$ where X is a v-set (of *points*) and $\mathcal{B}$ is a collection of subsets of X (called *blocks*) with sizes from K such that every pair of distinct points of X is contained in exactly one block of $\mathcal{B}$. The number $|X| = v$ is called the *order* of the PBD.

If Q^* denotes the set of all prime powers congruent to 0 or 1 modulo 7, then it is shown that a PBD $B(Q^*, 1; v)$ exists for all integers $v \geqslant 2136$, where $v \equiv 0$ or $1 \pmod 7$, with at most 104 possible exceptions below this value.

Before stating some fundamental results, we first wish to define the notion of resolvability of a $(v, k, 1)$-MD where, of course, $v(v-1) \equiv 0 \pmod k$.

Definition 1.4. If the blocks of a $(v, k, 1)$-MD for which $v \equiv 1 \pmod k$ can be partitioned into v sets each containing $(v-1)/k$ blocks which are pairwise disjoint (as sets), we say that the $(v, k, 1)$-MD is *resolvable* and any such partition is called a resolution of the design. Moreover, each set of $(v-1)/k$ pairwise disjoint blocks together with the singleton which is the only element not contained in any of its blocks is called a parallel class of the resolution. Any resolution of this kind has v parallel classes.

Definition 1.5. If the blocks of a $(v, k, 1)$-MD for which $v \equiv 0 \pmod k$ can be partitioned into $v-1$ sets each containing v/k blocks which are pairwise disjoint (as sets), we shall also say that the $(v, k, 1)$-MD is *resolvable* and each set of v/k pairwise disjoint blocks will be called a parallel class.

A $(v, k, 1)$-PMD which is resolvable in the sense of either Definition 1.4 or 1.5 will be denoted as a $(v, k, 1)$-RPMD.

Example 1.6. For $k=7$, we have the following examples of $(v, 7, 1)$-RPMDs where $v=7$ and $v=8$.

$$\begin{array}{ll}
(7,7,1)\text{-RPMD:} & X=\{1,2,3,4,5,6,7\},\ \mathcal{B} \text{ consists of} \\
& (1,2,3,4,5,6,7),\ (1,3,5,7,2,4,6), \\
& (1,4,7,3,6,2,5),\ (1,5,2,6,3,7,4), \\
& (1,6,4,2,7,5,3),\ (1,7,6,5,4,3,2).
\end{array}$$

$$\begin{array}{ll}
(8,7,1)\text{-RPMD:} & X=\{1,2,3,4,5,6,7,8\},\ \mathcal{B} \text{ consists of} \\
& (1,2,3,8,4,5,6),\ (1,3,5,7,6,4,8), \\
& (1,4,6,3,7,8,2),\ (1,5,4,2,8,6,7), \\
& (1,6,8,5,2,7,3),\ (1,7,2,6,5,3,4), \\
& (1,8,7,4,3,2,5),\ (2,4,7,5,8,3,6).
\end{array}$$

Constructions using finite fields and elementary abelian groups provide us with the following two useful results (see, for example, [4, 10, 11, 16]).

Theorem 1.7. *Let p be an odd prime and $r \geqslant 1$, then there exists a $(p^r, p, 1)$-PMD.*

Theorem 1.8. *Let $v=p^r$ be any prime power and $k>2$ be such that $k \mid (v-1)$, then there exists a $(v, k, 1)$-RPMD.*

The following two 'asymptotic' results can be found in [4, Theorem 4] and [16, Theorem 3.3].

Theorem 1.9. *A $(v, k, 1)$-RPMD exists for all sufficiently large v with $k \geqslant 3$ and $v \equiv 1 \pmod k$.*

Theorem 1.10. *A $(v, k, 1)$-PMD exists with $v(v-1) \equiv 0 \pmod k$ for the case when k is an odd prime and v is sufficiently large.*

We wish to remark that the term 'sufficiently large' appearing in Theorems 1.9 and 1.10 is unspecified, and the problem of finding a concrete bound for v in both cases remains to be solved in general. Evidently, from Theorem 1.10, we are guaranteed the existence of a constant C such that for all $v>C$, there exists a $(v, 7, 1)$-PMD where $v \equiv 0$ or $1 \pmod 7$. The results of this paper provides a concrete upper bound on C, namely, $C \leqslant 2135$.

2. Preliminaries

In this section, we shall define some terminology and state some fundamental results which will be used later. For more detailed information on PBDs and related designs, the interested reader may refer to [5, 9, 18, 19].

We shall denote by $B(K)$ the set of all integers v for which there exists a PBD $B(K, 1; v)$. For convenience, we write $B(k_1, k_2, \ldots, k_r)$ for $B(\{k_1, k_2, \ldots, k_r\})$. A set K is said to be *PBD-closed* if $B(K) = K$.

In what follows, we shall define

$$Q^* = \{q: q \equiv 0 \text{ or } 1 \pmod 7 \text{ and } q \text{ is a prime power}\},$$

$$\mathcal{P} = \{v: \text{there exists a } (v, 7, 1)\text{-PMD}\}.$$

The following result is contained in [16, Theorem 2.9].

Lemma 2.1. *Suppose* $v \in B(k_1, k_2, \ldots, k_r)$ *and for each* k_i *there exists a* $(k_i, k, 1)$-PMD. *Then there exists a* $(v, k, 1)$-PMD.

From Lemma 2.1 and Theorems 1.7 and 1.8, we can obtain the following useful lemma.

Lemma 2.2. $B(Q^*) \subset \mathcal{P}$.

Proof. From Theorems 1.7 and 1.8, we have $Q^* \subset \mathcal{P}$ and since $\mathcal{P}$ is PBD-closed by Lemma 2.1, we readily obtain $B(Q^*) \subset \mathcal{P}$. □

Definition 2.3. Let K and M be sets of positive integers. A *group divisible design* (GDD) $GD(K, 1, M; v)$ is a triple $(X, \mathcal{G}, \mathcal{B})$, where

(i) X is a v-set (of *points*),

(ii) $\mathcal{G}$ is a collection of non-empty subsets of X (called *groups*) with sizes in M and which partition X,

(iii) $\mathcal{B}$ is a collection of subsets of X (called *blocks*), each with size at least two in K,

(iv) no block meets a group in more than one point, and

(v) each pairset $\{x, y\}$ of points not contained in a group is contained in exactly one block.

The *group-type* (or *type*) of a GDD$(X, \mathcal{G}, \mathcal{B})$ is the multiset $\{|G|: G \in \mathcal{G}\}$ and we shall use the 'exponential' notation for its description: a group-type $1^i 2^j 3^k \cdots$ denotes i occurrences of groups of size 1, j occurrences of groups of size 2, and so on. A *weighting* of a GDD$(X, \mathcal{G}, \mathcal{B})$ is any mapping $w: X \to \mathbb{Z}^+ \cup \{0\}$.

Definition 2.4. A transversal design (TD) $T(k, 1; m)$ is a GDD with km points, k groups of size m and m^2 blocks of size k where each block meets every group in precisely one point, that is, each block is a transversal of the collection of groups.

Definition 2.5. Let $(X, \mathcal{B})$ be a PBD $B(K, 1; v)$. A *parallel class* in $(X, \mathcal{B})$ is a collection of disjoint blocks of $\mathcal{B}$, the union of which equals X. $(X, \mathcal{B})$ is called *resolvable* if the blocks of $\mathcal{B}$ can be partitioned into parallel classes. A GDD $GD(K, 1, M; v)$ is resolvable if its associated PBD $B(K \cup M, 1; v)$ is resolvable with M as a parallel class of the resolution.

It is fairly well-known that the existence of a resolvable TD $T(k, 1; m)$ (briefly $RT(k, 1; m)$) is equivalent to the existence of a $T(k+1, 1; m)$ or equivalently $k-1$ mutually orthogonal Latin squares (MOLS) of order m. Moreover, the following two results can be found in [12].

Theorem 2.6. *For every prime power* q, *there exists a* $T(q+1, 1; q)$.

Theorem 2.7. *Let* $m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ *be the factorization of* m *into powers of distinct primes* p_i, *then a* $T(k, 1; m)$ *exists, where* $k \leq 1 + \min\{p_i^{k_i}\}$.

We shall denote by $N(m)$ the maximum number of MOLS of order m. From current results we have on the existence of sets of 6 MOLS (see, for example, [8, 20]), we can state the following useful result.

Theorem 2.8. $N(m) \geq 6$ *for any* $m > 76$ *or for* $6 < m \leq 76$ *such that* m *is a prime power or* $m \in \{50, 56, 57, 63, 65, 69, 70, 72\}$. *For all such values of* m, *a* $T(8, 1; m)$ *exists.*

We shall briefly write $B(k, 1; v)$ for $B(\{k\}, 1; v)$ and similarly $GD(k, 1, m; v)$ for $GD(\{k\}, 1, \{m\}; v)$. We also observe that a PBD $B(k, 1; v)$ is essentially a *balanced incomplete block design* (BIBD) with parameters v, k, and $\lambda = 1$. If $k \notin K$, then $B(K \cup \{k^*\}, 1; v)$ denotes a PBD $B(K \cup \{k\}, 1; v)$ which contains a unique block of size k and if $k \in K$, then $B(K \cup \{k^*\}, 1; v)$ is a PBD $B(K, 1; v)$ containing at least one block of size k. We shall sometimes refer to a PBD $B(K, 1; v)$ as a $(v, K, 1)$-PBD, and a GDD$(X, \mathcal{G}, \mathcal{B})$ will be referred to as a K-GDD if $|B| \in K$ for every block B *in* $\mathcal{B}$. We shall also adapt the following notations:

$$B(k) = \{v\colon \text{a } (v, k, 1)\text{-BIBD exists}\},$$

$$RB(k) = \{v\colon \text{a resolvable } (v, k, 1)\text{-BIBD exists}\},$$

$$R_k = \{r\colon (k-1)r + 1 \in B(k)\},$$

$$R_k^* = \{r\colon (k-1)r + 1 \in RB(k)\},$$

It is known [19] that for any $k > 2$, the sets R_k and R_k^* are PBD-closed. Moreover, if k is a prime power, then it is known that $k+1 \in R_k^*$ and so we have $B(k+1) \subset R_k^*$. We also wish to point out that a $B(q+1, 1; q^2+q+1)$ is a projective plane of order q, denoted by $PG(2, q)$, and a $B(q, 1; q^2)$ is an affine plane of order q, denoted by $AG(2, q)$. An *oval* of an affine or projective plane of order q is a set of $q+1$ points, no three of which are collinear. The following useful result is proved in [5, VIII, 9.12 Theorem], and similar constructions can be found in [8, 18].

Theorem 2.9. *Let q be a prime power. Then there exists both an affine and a projective plane of order q with an oval. Hence for $0 \leq t \leq q+1$, we have*
(i) $q^2 - t \in B(q, q-1, q-2)$,
(ii) $q^2+q+1-t \in B(q+1, q, q-1)$.

The following result is due to Bose [6].

Theorem 2.10. *Let q be a prime power. Then $q^3+1 \in RB(q+1)$.*

We shall use a construction for PBDs due to Brouwer [7].

Theorem 2.11. *Suppose q is a prime power and $0 < t < q^2 - q + 1$. Then $t(q^2+q+1) \in B(t, q+t)$.*

The following construction is due to Seiden [17].

Theorem 2.12. *For any positive integer n, we have $2^{2n-1} - 2^{n-1} \in RB(2^{n-1})$.*

We shall also make use of the following result, which is contained in [13].

Theorem 2.13. *Let q be a prime power. Then $q^3+q^2+q+1 \in RB(q+1)$.*

For some of our recursive constructions of PBDs and GDDs, we shall make use of Wilson's 'Fundamental Construction' (see [19]). A brief description is presented below.

Construction 2.14 (Fundamental Construction). Suppose that $(X, \mathcal{G}, \mathcal{B})$ is a 'master' GDD and let $w : X \to \mathbb{Z}^+ \cup \{0\}$ be a weighting of the GDD. For every $x \in X$, let S_x be $w(x)$ copies of x. Suppose that for each block $B \in \mathcal{B}$, a GDD$(\bigcup_{x \in B} S_x, \{S_x : x \in B\}, \mathcal{A}_B)$ is given. Let $X^* = \bigcup_{x \in X} S_x$, $\mathcal{G}^* = \{\bigcup_{x \in G} S_x : G \in \mathcal{G}\}$, $\mathcal{B}^* = \bigcup_{B \in \mathcal{B}} \mathcal{A}_B$. Then $(X^*, \mathcal{G}^*, \mathcal{B}^*)$ is a GDD.

As already mentioned, our main result will be established on the basis of our investigation of the set $B(Q^*)$. It will be convenient for us to proceed in stages. Accordingly, we define the following sets:

$$D_0 = \{d : 7d \in B(Q^*)\},$$
$$D_1 = \{d : 7d + 1 \in B(Q^*)\},$$
$$D = D_0 \cap D_1.$$

Table 1

14	15	21	22	28	35	36	42	70	77
78	84	85	98	99	105	106	126	133	134
140	141	147	148	154	155	161	162	168	175
176	182	183	189	190	196	217	218	224	231
238	245	246	252	253	259	260	266	267	273
274	280	287	288	294	295	315	316	322	329
364	371	378	420	574	581	582	588	609	616
623	630	665	1078	1085	1086	1092	1099	1106	1107
1113	1114	1120	1127	1134	1162	1169	1170	1176	1197
1198	1218	1253	1260	1351	1372	1393	1400	1407	1414
2037	2044	2093	2135						

Since D is not PBD-closed, we shall consider the following set $U \subset D$ which is PBD-closed and defined by:

$U = \{u\colon \text{there exists a } GD(B(Q), 1, 7; 7u)\}$,

where $Q = \{7, 8, 29, 43, 169, 197, 239, 281\} \subset Q^*$.

In what follows, we shall first investigate the sets U, D_1 and D_0, and then apply Lemma 2.2 to obtain the following main result:

Main Theorem 2.15. *For every positive integer* $v \equiv 0$ *or* 1 (mod 7), *with the possible exception of the* 104 *values shown in Table 1, there exists a* $(v, 7, 1)$-PMD.

3. Determination of U

The following lemma is fairly obvious.

Lemma 3.1. *If* $u \in B(Q)$ *and* $N(u) \geqslant 6$, *then* $u \in U$. *In particular,* $Q \subset U$.

Lemma 3.2. $\{9, 13\} \subset U$.

Proof. In a $T(8, 1; 8)$, we delete one point to obtain a $GD(8, 1, 7; 63)$ and $9 \in U$. In [9], the existence of a $GD(7, 1, 7; 91)$ is shown. Hence $13 \in U$. □

Lemma 3.3. $189 \in U$.

Proof. Apply Theorem 2.11 with $t = 9$ and $q = 4$ to obtain $189 \in B(9, 13)$. From Lemma 3.2 and the fact that U is PBD-closed, we get $189 \in U$. □

Lemma 3.4. *If* $v \in B(8)$, *then* $(v - e)/7 \in U$ *where* $e = 1$ *or* 8.

Proof. In a $(v, 8, 1)$-BIBD, we delete e points from a particular block to obtain a $GD(\{7, 8\}, 1, 7; v - e)$. The conclusion follows. □

Lemma 3.5. $\{120, 344\} \subset RB(8)$.

Proof. Apply Theorem 2.12 with $n = 4$ to obtain $120 \in RB(8)$ and apply Theorem 2.10 with $q = 7$ to get $344 \in RB(8)$. $\square$

As an immediate consequence of Lemmas 3.4 and 3.5, we obtain

Corollary 3.6. $\{16, 17, 48, 49\} \subset U$.

Lemma 3.7. *If $v \in RB(7)$, then $v/7 + u \in U$, where $u \in U$ and $7u < (v-1)/6$.*

Proof. We adjoin $7u$ infinite points to a resolvable $(v, 7, 1)$-BIBD, where one infinite point is adjoined to each of $7u$ parallel classes of blocks. In the resulting design, we then take as groups the blocks of one of the remaining parallel classes together with the block at infinity of size $7u$ to obtain a $\{7, 8\}$-GDD of group-type $7^{v/7}(7u)^1$. Since $u \in U$, we can then break up the group of size $7u$ to form a $GD(B(Q), 1, 7; 7u)$ and obtain $v/7 + u \in U$. $\square$

Corollary 3.8. $\{55, 56, 62, 110, 116, 295, 296, 304, 312\} \subset U$.

Proof. Since $\{64, 120, 344\} \subset B(8) \subset R_7^*$, we have $\{385, 721, 2065\} \subset RB(7)$. We then apply Lemma 3.7 with $u = 0, 1, 7, 9, 13$, or 17 to obtain the desired result. $\square$

Lemma 3.9. *If $N(t) \geq 6$ and $t + q \in B(Q)$ where $q = 0$ or 1, then $t + u \in U$ provided $u \in U$ and $7u < t$.*

Proof. We take an $RT(7, 1; t)$ and adjoin $7u - q$ infinite points to $7u - q$ parallel classes of blocks and $q = 0$ or 1 infinite point to the groups so as to form a $\{7, 8, t + q\}$-GDD of group-type $7^t(7u)^1$. Using the fact that $t + q \in B(Q)$ and $u \in U$, we can then break up the size $t + q$ blocks and the group of size $7u$ to form a $B(Q)$-GDD of group-type $7^t 7^u$ and obtain $t + u \in U$. $\square$

Corollary 3.10. $\{30, 44, 50, 51, 57, 58, 170, 176, 177, 178, 182, 185, 186, 198, 214, 252, 268, 282, 287, 290, 293, 294, 297, 298, 349, 354, 370\} \subset U$.

Proof. For most values, we apply Lemma 3.9 with $q = 0$, $t = 29, 43, 50, 57, 169, 197, 239, 281, 336, 337, 357$ and appropriate values of u already determined. Since $\{7, 48, 51\} \subset U$, we know that $\{50, 336, 337, 357\} \subset B(Q)$. It is easy to see that $57 \in B(Q)$. For $\{287, 293\} \subset U$, we apply Lemma 3.9 with $q = 1$, $t = 280$ and $u = 7, 13$. $\square$

The following lemma is an immediate consequence of Theorem 2.9.

Lemma 3.11. *If $63 \leq u \leq 81$, then $u \in U$.*

Proof. We apply Theorem 2.9 with $q = 9$ in (i) and $q = 8$ in (ii) to obtain $u \in B(7, 8, 9)$ for $63 \leqslant u \leqslant 81$. Since $\{7, 8, 9\} \subset U$ and U is PBD-closed, the conclusion follows. □

Lemma 3.12. *If $N(t) \geqslant 6$ and $t + 7 \in B(Q \cup \{7^*\})$, then $t + 1 \in U$.*

Proof. We take an $RT(7, 1; t)$ and adjoin 7 infinite points to it by forming a $(t + 7, Q \cup \{7^*\}, 1)$-PBD on each group in such a way that the 7 infinite points become a common block in these PBDs. Taking a parallel class of blocks in the $RT(7, 1; t)$ together with the block at infinity of size 7 as groups we obtain a $B(Q)$-GDD of group-type 7^{t+1}. That is, $t + 1 \in U$.

Corollary 3.13. $\{85, 86\} \subset U$.

Proof. Take $t = 84$, 85 in Lemma 3.12. The conditions $91 \in B(7)$ and $92 \in B(7, 8)$ come from the existence of a $GD(7, 1, 7; 91)$ in [9]. □

Lemma 3.14. *Suppose $t \in U$ and t is a prime power greater than 8. Suppose $7 + b \in U$ and $7 + b \leqslant t$. Then $7t + a + b \in U$ if $a \in U$ and $0 \leqslant a < t$; or $7t + a + b - 1 \in U$ if $a \in U$ and $0 < a \leqslant t$.*

Proof. In a $T(t + 1, 1; t)$, we take a particular block B of size $t + 1$. Keeping a points in the eighth group of the TD, we delete the other points including the point of intersection with B. Keeping the points in the first seven groups of the TD and b additional points in B, we delete all the other points in the last $t - 7$ groups so as to obtain a $(7t + a + b, U, 1)$-PBD, that is, $7t + a + b \in U$. If we leave the point of intersection of B and the eighth group undeleted, then we get $7t + a + b - 1 \in U$. □

Lemma 3.15. *Suppose $N(t) \geqslant 7$ and $q \in \{0, 1\}$. Then $\{t + q, u + q, v + q\} \subset U$ implies $7t + u + v + q \in U$.*

Proof. In a $T(9, 1; t)$ we delete $t - u$ points from one group and $t - v$ points from another group to obtain a $\{7, 8, 9\}$-GDD of group-type $t^7u^1v^1$. We then adjoin q infinite points to the groups of this GDD in order to obtain the desired result. □

Lemma 3.16. *If $91 \leqslant n \leqslant 145$ and $n \neq 95$, 96, then $n \in U$.*

Proof. We apply Lemma 3.15 with $t = 13, 16, 17$ to get $7t + u + v + q \in U$, where the required parameters are shown in *Table* 2. In addition, we can take $(a, b) \in \{(1, 2), (0, 6), (9, 2), (7, 6)\}$ in Lemma 3.14 to obtain $7t + m \in U$ where $m = 3, 6, 11, 12$. In particular, we have $7 \cdot 13 + m \in U$ for $0 \leqslant m \leqslant 3$ and $6 \leqslant m \leqslant 18$. We also apply Lemma 3.14 with $t = 17$ and $(a, b) = (13, 6)$ and, in

Table 2

$u+v+q$	u	v	q
0	0	0	0
1	0	1	0
2	1	1	0
3			
4			
5			
6			
7	0	7	0
8	1	7	0
9	1	8	0
10	1	9	0
11			
12			
13	0	13	0

Table 2 (contd.)

$u+v+q$	u	v	q
14	7	7	0
15	7	8	0
16	8	8	0
17	8	9	0
18	9	9	0
19	6	12	1
20	7	13	0
21	8	13	0
22	9	13	0
23	7	16	0
24	8	16	0
25	9	16	0
26	13	13	0

conjunction with *Table 2*, we then have $7 \cdot 17 + m \in U$ for $6 \leqslant m \leqslant 26$. All the values of n have been covered except for $n = 110$ and $n = 116$, which are taken care of by Corollary 3.8.

Definition 3.17. An *oval in a group divisible design* is a set of points which intersects each group in at most one point and each block in at most two points.

Lemma 3.18. *If* $146 \leqslant n \leqslant 153$, *then* $n \in U$.

Proof. It is clear that there is an oval of size 7 in a $T(9, 1; 17)$. By deleting k points from the oval, we get $153 - k \in B(7, 8, 9, 16, 17) \subset U$ where $0 \leqslant k \leqslant 7$. □

Lemma 3.19. $n \in U$ *if* $203 \leqslant n \leqslant 300$ *and* $n \notin E_1$ *where* $E_1 = \{207, 208, 250, 251, 266, 267, 271, 274, 275, 276, 277, 278, 283, 284, 286, 291, 292, 299, 300\}$.

Proof. We apply Lemma 3.15 with $t = 29$ and $q \in \{0, 1\}$ to get $7t + u + v + q \in U$ where the required parameters are shown in *Table 2* and *Table 3*. We can also

Table 3

$u+v+q$	u	v	q
27			
28	12	15	1
29	13	16	0
30	13	17	0
31	15	15	1
32	16	16	0
33	16	17	0
34	17	17	0
35	6	28	1
36	7	29	0

Table 3 (contd.)

$u+v+q$	u	v	q
37	8	29	0
38	9	29	0
39	9	29	1
40			
41	12	28	1
42	13	29	0
43	13	29	1
44	15	28	1
45	16	29	0
46	16	29	1

apply Lemma 3.14 to show that $7t + m \in U$ for $m = 3$, 6, 11, 12 and 27 where for $m = 27$, we take $a = 17$ and $b = 10$. This takes care of the cases where $203 \leqslant n \leqslant 249$ except $n = 243$. For $n = 243$, we take a $T(9, 1; 29)$ and delete 14 points in the last group. It is easy to see that there is an oval of size 6 in the resulting GDD. Adding a new point to each group of this GDD and deleting 5 points of the oval, we get $243 \in U$. From Corollary 3.10 we have $252 \in U$. In a $T(9, 1; 29)$ there is an oval of size 9, and appropriate addition and deletion of points will yield $n \in U$ for $253 \leqslant n \leqslant 262$. For the stated values of $n \in U$ where $263 \leqslant n \leqslant 289$, except for $n = 268$, 270, 282, 287, we take either a $T(17, 1; 16)$ or a $T(17, 1; 17)$ and appropriately delete some points from one group so that the resulting group sizes are all in U. However, $270 \in U$ follows from deleting a block from a $T(9, 1; 31)$ and $\{268, 282, 287\} \subset U$ from Corollary 3.10. Finally, the remaining cases for $290 \leqslant n \leqslant 300$ are covered in Corollaries 3.7 and 3.10. □

We are now in the position to prove the following theorem.

Theorem 3.20. *$n \in U$ for every positive integer n, with the possible exception of those values listed in Table 4.*

Proof. From our previous lemmas and corollaries, we know that the conclusion holds for $n \leqslant 300$. Applying Lemmas 3.14 and 3.15 with $t = 43$, 49, 56, 64, 67, 73, 79 and the parameters shown in *Tables 2, 3,* and *5*, we obtain that the conclusion holds for $n \leqslant 614$, where for $7t + 40$ we let $a = 30$ and $b = 10$ in Lemma 3.14. When $t \geqslant 81$ and $62 \leqslant u \leqslant 81$, we can choose $q = 0$ and $v \leqslant 81$ such that $7t + u + v + q \in U$ and $62 \leqslant u + v + q \leqslant 162$. Consequently, we obtain $n \in U$ for $573 \leqslant n \leqslant 2339$ as illustrated in *Table 6*, where the required $T(q, 1; t)$ comes from [8]. We now apply Lemma 3.15 recursively for each $t \geqslant 307$ and $q = u = 0$, $62 \leqslant v \leqslant 81$ to get $7t + v \in U$. Since $u = 0$, we only need the condition $N(t) \geqslant 6$ which can be obtained from Theorem 2.8. This guarantees that $n \in U$ whenever $n \geqslant 2211$, and the proof of the theorem is complete. □

Table 4

2	3	4	5	6	10	11	12	14	15
18	19	20	21	22	23	24	25	26	27
28	31	32	33	34	35	36	37	38	39
40	41	42	45	46	47	52	53	54	**59**
60	**61**	82	83	84	87	88	89	90	95
96	154	155	156	157	158	159	160	161	162
163	**164**	**165**	166	167	168	171	**172**	**173**	174
175	179	180	**181**	**183**	**184**	**187**	**188**	**190**	**191**
192	193	**194**	**195**	196	199	200	201	202	**207**
208	**250**	**251**	**266**	**267**	**271**	**274**	**275**	**276**	**277**
278	**283**	**284**	**286**	291	292	299	**300**	305	**306**

Table 5

$u+v+q$	u	v	q
47	17	30	0
48	0	48	0
49	6	42	1
50	7	43	0
51	8	43	0
52	9	43	0
53	9	44	0
54	6	47	1
55	12	42	1
56	13	43	0
57	28	28	1
58	29	29	0
59	29	29	1
60	30	30	0
61	17	44	0

Table 6

t	$7t+6 \leq n \leq 7t+162$
81	573 - 729
91	643 - 799
107	755 - 911
129	909 - 1065
151	1063 - 1219
169	1189 - 1345
185	1301 - 1457
205	1441 - 1597
227	1595 - 1751
248	1742 - 1898
269	1889 - 2045
289	2029 - 2185
311	2183 - 2339

4. Determination of D_1

From some prime powers $q \equiv 1 \pmod 7$ in Q^* or a product of two numbers in Q^*, we obtain

Lemma 4.1. $\{4, 6, 10, 18, 24, 28, 34, 40, 54, 60, 88, 90, 96, 156, 166, 184, 190, 196, 276, 306\} \cup \{33\} \subset D_1$.

Lemma 4.2. *If there exists a $GD(U, 1, D_1; d)$, then $d \in D_1$.*

Proof. In the $GD(U, 1, D_1; d)$, we give weight 7 to every point. Since a $GD(B(Q), 1, 7; 7u)$ exists for every block of size $u \in U$, Construction 2.14 gives us a GDD with block sizes all in $B(Q)$ and group sizes of the form $7k$ where $k \in D_1$. We then adjoin one infinite point to this GDD to obtain a $(7d+1, B(Q^*), 1)$-PBD, and the result follows. □

Corollary 4.3. $\{82, 84, 87, 89, 95\} \subset D_1$.

Proof. We shall apply Lemma 4.2 as follows. In a $T(9, 1; 11)$, we consider two disjoint blocks and delete all but one of the points to obtain $82 \in D_1$. If we delete one block and some points in a group of a $T(9, 1; 11)$ in such a way that the small group is of size 4, 7, or 9, then we get $\{84, 87, 89\} \subset D_1$. In a $T(9, 1; 13)$ we delete one group entirely and delete 9 points from a second group to obtain $95 \in D_1$. □

Corollary 4.4. $\{154, 157, 160, 161, 162\} \subset D_1$.

Proof. In a $T(9, 1; 19)$ we delete one block and some additional points from a group. □

Corollary 4.5. $\{192, 193, 200, 201, 202, 208\} \subset D_1$.

Proof. In a $T(9, 1; 25)$, we delete one block and some additional points from a group. □

Corollary 4.6. $\{207, 250, 251\} \subset D_1$.

Proof. In a $T(9, 1; 29)$ we delete one group entirely and further delete 25 points from another group to obtain $207 \in D_1$. If we delete 5 points from each of two groups in a $T(9, 1; 29)$, we obtain $251 \in D_1$. For $250 \in D_1$ we delete one block and 20 points in a group of a $T(9, 1; 31)$. □

Corollary 4.7. $305 \in D_1$.

Proof. In a $T(8, 1; 43)$, we delete 39 points from one group. □

Lemma 4.8. *If $N(t) \geq 6$ and $t + q \in B(Q^*)$ where $q = 0$ or 1, then $t + e \in D_1$ provided that $e \in D_1$ and $7e + 1 \leq t + q$.*

Proof. In a $T(8, 1; t)$ we delete some points from a group so that the group size is $7e + 1 - q$. We then adjoin q infinite points to each group of the resulting GDD to get a $(7t + 7e + 1, \{7, 8, t + q, 7e + 1\}, 1)$-PBD and $t + e \in D_1$. □

Corollary 4.9. $\{47, 53, 168, 172, 173, 175, 179, 187, 266, 267, 284, 286, 291, 299\} \subset D_1$.

Proof. We apply Lemma 4.8 with the parameters shown in *Table 7*, where $N(t) \geq 6$ and $t + q \in B(Q^*)$. □

Table 7

$t+e$	t	e	q
47	43	4	0
53	49	4	1
168	168	0	1
172	168	4	1
173	169	4	0
175	169	6	0
179	169	10	0
187	169	18	0
266	238	28	1
267	238	29	1
284	280	4	1
286	280	6	1
291	281	10	0
299	281	18	0

Lemma 4.10. *If $v \in RB(7)$, then $v/7 + e \in D_1$ provided that $e \in D_1$ and $7e + 1 \leq (v - 1)/6$.*

Proof. We adjoin $7e + 1$ infinite points to a resolvable $(v, 7, 1)$-BIBD, where one infinite point is adjoined to each of $7e + 1$ parallel classes of blocks. We obtain $v + 7e + 1 \in B(7, 8, 7e + 1) \subset B(Q^*)$ and the conclusion follows. □

Corollary 4.11. $\{59, 61\} \subset D_1$.

Proof. There is a resolvable $(385, 7, 1)$-BIBD as shown in the proof of Corollary 3.8. We then apply Lemma 4.10 with $e = 4$ and 6 to get the desired result. □

Lemma 4.12. *If $t \in B(Q^*)$ is a prime power, then $t + e \in D_1$ provided $e + 1 \in D_1$ and $7e + 7 \leq t$.*

Proof. In a $T(t + 1, 1; t)$, we delete all the points in $t - 6$ groups except for $7e + 1$ points which lie in the same block. This gives $7t + 7e + 1 \in B(7, 8, t, 7e + 8) \subset B(Q^*)$ and $t + e \in D_1$. □

Corollary 4.13. $\{32, 46, 52\} \subset D_1$.

Proof. We apply Lemma 4.12 with $t \in \{29, 43, 49\}$ and $e = 3$. □

Lemma 4.14. $\{271, 274, 277, 278, 283\} \subset D_1$.

Proof. In a $T(9, 1; 32)$ we delete all but a points in one group and delete all but b points in another group to obtain a $GD(\{7, 8, 9\}, 1, \{32, a, b\}; 224 + a + b)$. We then apply Lemma 4.2 with $(a, b) \in \{(30, 17), (32, 18), (29, 24), (30, 24), (30, 29)\}$ to obtain the desired result. □

The following lemma is a more general form of Lemma 4.8.

Lemma 4.15. *Suppose $t + q \in B(Q^* \cup \{q^*\})$ and $N(t) \geq 6$. Then $t + e \in D_i$ provided $e \in D_i$ and $7e + i - q \leq t$, where $i \in \{0, 1\}$.*

Proof. In a $T(8, 1; t)$, we delete some points from one group so that the truncated group size is $7e + i - q$. We then adjoin q infinite points to the groups of the resulting GDD so as to form a $(7t + 7e + i, B(Q^*), 1)$-PBD and get $t + e \in D_i$ for $i = 0$ or 1. □

Corollary 4.16. $\{163, 164, 165, 166, 174, 180, 181, 183, 188, 191, 194, 195, 199, 275, 292, 300\} \subset D_1$.

Table 8

$t+e$	t	e	q
163	156	7	31
164	156	8	31
165	156	9	31
166	156	10	31
174	156	18	31
180	156	24	31
181	168	13	1
183	174	9	29
188	175	13	29
191	181	10	29
194	181	13	29
195	182	13	29
199	175	24	29
275	258	17	43
292	258	34	43
300	287	13	43

Proof. We apply Theorem 2.13 with $q = 5$ to obtain $156 \in RB(6)$. The introduction of 31 infinite points to a resolvable $(156, 6, 1)$-BIBD yields $187 \in B(7, 31^*)$. On the other hand, it is easy to see that the following hold:

$$169 \in B(Q^*),$$
$$203 = 7 \cdot 29 \in B(7, 29),$$
$$204 = 7 \cdot 29 + 1 \in B(7, 8, 29),$$
$$210 = 7 \cdot 29 + 7 \in B(7, 8, 29),$$
$$211 = 7 \cdot 29 + 8 \in B(7, 8, 29),$$
$$301 = 7 \cdot 43 \in B(7, 43),$$
$$302 = 7 \cdot 43 + 1 \in B(7, 8, 43),$$
$$309 = 7 \cdot 43 + 8 \in B(7, 8, 43),$$
$$330 = 7 \cdot 43 + 29 \in B(7, 8, 29, 43).$$

We then apply Lemma 4.15 with $i = 1$ and the parameters shown in *Table 8* in order to obtain the desired result. □

Summarizing the above results, we have proved

Theorem 4.17. *$n \in D_1$ for every positive integer n, with the possible exception of those values listed in Table 9.*

Table 9

2	3	5	11	12	14	15	19	20	21
22	23	25	26	27	31	35	36	37	38
39	41	42	45	83	155	158	159	167	171

5. Determination of D_0

Lemma 5.1. *There exist* $\{7, 8\}$-GDDs *of the following group-types*: (a) 7^7, (b) 7^8, (c) 7^9, (d) 6^8, (e) $6^8 7^1$, (f) $7^7 6^1$, (g) $7^8 6^1$.

Proof. (a), (b) are fairly obvious. For (c) and (d), we delete one point from a $B(8, 1; 64)$ and a $B(7, 1; 49)$, respectively. For (e), we delete one point from a $T(7, 1; 8)$. For (f), we delete one point from a $T(8, 1; 7)$. For (g), we delete two points from a group in a $T(8, 1; 8)$. □

Lemma 5.2. *Suppose* $N(t) \geq 7$ *and* $6t + q \in B(Q^* \cup \{q^*\})$. *Then* $48t + 7u + q \in B(Q^*)$ *provided* $7u + q \in B(Q^*)$ *and* $u \leq t$.

Proof. In all groups but one of a $T(9, 1; t)$, give the points weight 6. In the last group, we give u points weight 7 and give the remaining points weight 0. We can apply Construction 2.14 with the necessary input designs from Lemma 5.1 to obtain a $\{7, 8\}$-GDD of group-type $(6t)^8(7u)^1$. We then adjoin a set of q infinite points to the groups of this GDD, using the fact that $6t + q \in B(Q^* \cup \{q^*\})$ and $7u + q \in B(Q^*)$ to obtain the desired result. □

Corollary 5.3. $\{59, 61, 207\} \subset D_0$.

Proof. We apply Lemma 5.2 with the parameters shown in *Table 10.* Note that we make use of the fact that $203 \in B(7, 29)$ as shown in the proof of Corollary 4.16. □

Lemma 5.4. $\{163, 164, 165, 172, 173, 175, 181, 183, 184, 187, 188, 190, 191, 192, 194, 195, 266, 267, 271, 274, 275, 276, 300\} \subset D_0$.

Proof. The proof is similar to that of Corollary 4.16. We shall apply Lemma 4.15 with $i = 0$ and the parameters shown in *Table 11,* where $e \in U \subset D_0$. □

Lemma 5.5. *Suppose* $N(t) \geq 7$ *and* $\{u, t\} \subset D_1$, *where* $u \leq t$. *Suppose* $e \in D_0$ *such that* $7e = 7x + 6y + 1$ *where* $0 \leq x + y \leq t$. *Then* $7t + u + e \in D_0$.

Proof. In all groups but two of a $T(9, 1; t)$, we give the points weight 7. In the second last group, we give u points weight 7 and give the remaining points weight

Table 10

d	$7d$	t	$7u$	q
59	413	8	28	1
61	427	8	42	1
207	1449	29	28	29

Table 11

$t+e$	t	e	q
163	156	7	31
164	156	8	31
165	156	9	31
172	156	16	31
173	156	17	31
175	168	7	1
181	168	13	1
183	174	9	29
184	168	16	1
187	174	13	29
188	175	13	29
190	174	16	29
191	174	17	29
192	175	17	29
194	181	13	29
195	182	13	29
266	258	8	43
267	258	9	43
271	258	13	43
274	258	16	43
275	258	17	43
276	259	17	43
300	287	13	43

0. In the last group, we give weight 7 to x points, weight 6 to y points and give the remaining points weight 0. We can apply Construction 2.14 with the necessary input designs from Lemma 5.1 to obtain a $\{7, 8\}$-GDD of group-type $(7t)^7(7u)^1(7x+6y)^1$. Adjoining one infinite point to the groups of this GDD and using the fact that $\{u, t\} \subset D_1$, we get $7t+u+e \in D_0$, where $7e = 7x+6y+1$. □

Corollary 5.6. $\{96, 208, 250, 251, 277, 278, 283, 284, 286, 306\} \subset D_0$.

Table 12

$7t+u+e$	t	u	e	x	y
96	13	4	1	0	1
208	29	4	1	0	1
250	29	18	29	28	1
251	32	18	9	8	1
277	32	24	29	28	1
278	32	24	30	29	1
283	32	29	30	29	1
284	32	30	30	29	1
286	32	32	30	29	1
306	43	4	1	0	1

Proof. We apply Lemma 5.5 with the parameters shown in *Table 12*. □

Summarizing the results of this section, we have essentially proved

Theorem 5.7. $n \in D_0$ for every positive integer n, with the possible exception of those values not bold-faced in *Table 4.*

6. Conclusion

Combining the results of Theorems 4.17, and 5.7, we obtain

Theorem 6.1. *Let* Q^* *denote the set of all prime powers* $q \equiv 0$ *or* 1 (mod 7). *Then* $v \in B(Q^*)$ *for all positive integers* $v \equiv 0$ *or* 1 (mod 7), *with the possible exception of the* 104 *values listed in Table* 1.

Proof of Main Theorem 2.15. Since $B(Q^*) \subset \mathcal{P}$, the result is an immediate consequence of Theorem 6.1. □

Remark. The problem of existence of (v, k, λ)-PMDs has been completely settled for the case $k = 3$ (see [1, 15]). However, for $k > 3$ the problem is still open and currently under investigation. Further results on the cases $k = 4, 5, 6$ will be reported in subsequent papers.

Note added in proof

Since this paper was accepted for publication, the authors would like to mention the following improvements and current results:

(1) It is now known that a $(v, 7, 1)$-PMD exists for all integers $v \geqslant 421$, where $v \equiv 0$ or 1 (mod 7), with at most 40 possible exceptions below this value.

(2) It has been determined that $v \in B(Q^*)$ holds for all integers $v \geqslant 1415$, where $v \equiv 0$ or 1 (mod 7), with at most 84 possible exceptions below this value.

(3) An almost complete solution has now been obtained to the problem of existence of (v, k, λ)-PMDs for the cases $k = 4$ and $k = 5$.

References

[1] F.E. Bennett, Direct constructions for perfect 3-cyclic designs, Ann. Discrete Math. 15 (1982) 63–68.

[2] F.E. Bennett, Conjugate orthogonal Latin squares and Mendelsohn designs, Ars. Combin. 19 (1985) 51–62.

[3] F.E. Bennett, On r-fold perfect Mendelsohn designs, Ars. Combin. 23 (1987) 57–68.
[4] F.E. Bennett, E. Mendelsohn and N.S. Mendelsohn, Resolvable perfect cyclic designs, J. Combin. Theory Ser. A 29 (1980) 142–150.
[5] Th. Beth, D. Jungnickel and H. Lenz, Design Theory, Bibliographisches Institut, Zurich (1985).
[6] R.C. Bose, On the application of finite projective geometry for deriving a certain series of balanced Kirkman arrangements, Calcutta Math. Soc., Golden Jubilee Volume (1959) 341–354.
[7] A.E. Brouwer, A series of separable designs with application to pairwise orthogonal Latin squares, European. J. Combin. 1 (1980) 39–41.
[8] A.E. Brouwer, The number of mutually orthogonal Latin squares—a table up to order 10 000, Research Report ZW123/79, Mathematisch Centrum, Amsterdam, 1979.
[9] H. Hanani, Balanced incomplete block designs and related designs, Discrete Math. 11 (1975) 255–369.
[10] D.F. Hsu and A.D. Keedwell, Generalized complete mappings, neofields, sequenceable groups and block designs. II, Pacific J. Math. 117 (1985) 291–312.
[11] A.D. Keedwell, Circuit designs and Latin squares, Ars. Combin. 17 (1984) 79–90.
[12] H.F. MacNeish, Euler squares, Ann. Math. 23 (1922) 221–227.
[13] R. Mathon, On the existence of doubly resolvable Kirkman systems and equidistant permutation arrays, Discrete Math. 30 (1980) 157–172.
[14] N.S. Mendelsohn, Combinatorial designs as models of universal algebras, in: Recent Progress in Combinatorics (Academic Press, New York 1969) 123–132.
[15] N.S. Mendelsohn, A natural generalization of Steiner triple systems, in: A.O.L. Atkin and B.J. Birch, eds., Computers in Number Theory (Academic Press, New York, 1971) 323–338.
[16] N.S. Mendelsohn, Perfect cyclic designs, Discrete Math. 20 (1977) 63–68.
[17] E. Seiden, A method of construction of resolvable BIBD, Sankhyā (A) 25 (1963) 393–394.
[18] J.H. Van Lint, Combinatorial Theory Seminar Eindhoven University of Technology. Lecture Notes in Math. 382 (Springer, New York, 1974).
[19] R.M. Wilson, Constructions and uses of pairwise balanced designs, Math. Centre Tracts 55 (1974) 18–41.
[20] L. Zhu, Six pairwise orthogonal Latin squares of order 69, J. Austral. Math. Soc. Ser. A 37 (1984) 1–3.